Discrete Burridge-Knopoff model, with exact solitonic or compactlike traveling wave solution

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We have explored the dynamics of two versions of a Burridge-Knopoff model: with linear or nonlinear interactions between adjacent blocks. We have shown that by properly choosing the analytical form of the discrete solitary wave solution of the model we can calculate analytically the form of the friction function. In both cases our analytical results show that the friction force naturally presents the behavior of a simple weakening friction law first introduced qualitatively by Burridge and Knopoff [Bull. Seismol. Soc. Am. **57**, 3411 (1967)] and quantitatively by Carlson and Langer [Phys. Rev. Lett. **62**, 2632 (1989)]. With such a force function the discrete solitonic or compactlike wave-front solutions are exact and stable solutions. In the case of linear coupling our numerical simulations show that an irregular initial state evolves into kink pairs (large-amplitude events), that can recombine or not, plus nonlinear localized modes and small linear oscillations (small-amplitude events) that disperse with time, owing to dispersion. For nonlinear coupling one observes compactlike kink pairs or shocks, and a background of robust incoherent nonlinear oscillations (small amplitude events) that persist with time. Our results show that discreteness is a necessary ingredient to observe a rich and complex dynamical behavior. Nonlinearity allows the existence of strictly localized shocks.

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I. INTRODUCTION

Recently, excitable media with elastic rather than diffusive coupling between spatial elements have been investigated [1,2]. Among the systems of special interest are the elastic media with friction, which exhibit stick-slip motion. In this context the discrete spring-block model [3], originally introduced by Burridge and Knopoff (BK) (1967) and intensively studied by Carlson and Langer (1989) [4], has regained considerable attention since Bak and Tang [5] suggested that crustal faults may exhibit a phenomenon known as "self-organized criticality." The idea is that many systems in nature, a sand pile, an avalanche or the earth's crust for example, are driven by external forces in such a way that they are always at or near a threshold of instability and might be expected to exhibit anomalously large "critical" fluctuations.

The standard BK model, which was originally introduced in order to reproduce the gross features of the statistics of real earthquakes, describes the dynamics of a slowly driven slider-block chain in the presence of a nonlinear friction exhibiting a velocity softening instability. The friction law is an essential ingredient of this dynamical behavior. In this context, numerous slider-block models have been studied (see [6] for a review and [7] for an extensive reference list). They have been used to explore the role of friction along a fault as a factor in the earthquake dynamics and are capable of supporting steadily propagating solitary waves into the form of shocks [8–11]. Very recently, a continuum version of the BK system with an asymptotically velocity strengthening friction model [1] or with a Coulomb friction model [2] were studied as excitable media. It was suggested that this type of elastic medium may have applications beyond laboratory friction experiments to electronic transmission lines and active optical waveguides [1].

As far as we know, in all these approaches different versions of the friction law were postulated and no analytical study of the discrete BK model has been performed. As the friction law represents the key nonlinearity leading to complex behavior, and knowing that discreteness effects may play a role [12] in the spatiotemporal behavior of earthquake occurrence, it is extremely important to develop the basic concepts with the help of simple models.

The purpose of this paper is to make some progress in the understanding of the effects of discreteness and nonlinear interactions, respectively, on the dynamical behavior of a one-dimensional slider-block model. The paper is organized as follows: In Sec. II we investigate a discrete BK model where the blocks execute linear interactions. In Sec. III, we examine the case of nonlinear interactions. For both cases we construct exact kink or wave-front solutions, that is contrary to all the previous investigations where a friction model has been assumed, we adopt an inverse procedure (see [13] and references therein). Indeed, we show that by properly choosing the analytical expression of the discrete solitary wave solution of the model we can analytically calculate the form of the friction function. Since in a real system some degree of heterogeneity should be present, we next explore by numerical simulations the complex dynamics of the model resulting from an irregular initial state. Finally, in the last section we give some concluding remarks.

II. BK MODEL WITH LINEAR COUPLING

The BK model describes the contact region between two tectonic plates. It consists (as schematically represented in Fig. 1) of a one-dimensional array of N identical blocks of

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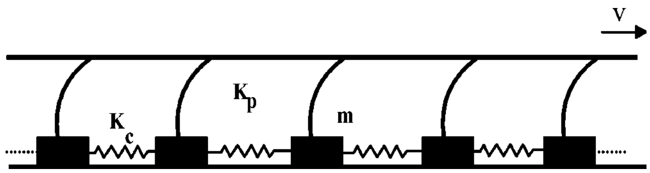

FIG. 1. Mechanical Burridge-Knopoff earthquake model. It consists of a chain of mass m coupled by springs of stiffness K_c, linked to an upper plate by springs of stiffness K_p and pulled at constant velocity V against a nonlinear frictional force F.

mass m coupled by springs of stiffness k_c and equilibrium length a. Each block is linked to the upper plate by a spring of stiffness k_p. The blocks resting on the frictional surface of the lower plate are pulled by the upper plate, which moves at constant (driving tectonic) velocity V, against the frictional force F. If X_n is the displacement of the nth block, the equation of motion reads as

$$m\frac{d^2X_n}{dt^2}=k_c[(X_{n-1}-X_n)^q+(X_{n+1}-X_n)^q]-k_p(X_n-Vt)-F(\dot{X}_n), \tag{1}$$

where $\dot{X}_n$ is the velocity at site n. When parameter q is equal to 1, we have the standard BK model with linear coupling that we will study in the present section.

Let us introduce $X_n=X_0\psi_n$, where X_0 is a constant, and setting $\omega_p=\sqrt{k_p/m}$, $t'=\omega_p t$, $v=V/X_0\omega_p$, $\gamma=1/k_pX_0$, and $K=k_c/k_p$ we obtain the following dimensionless equation for ψ_n:

$$\frac{d^2\psi_n}{dt'^2}=K(\psi_{n+1}-2\psi_n+\psi_{n-1})-(\psi_n-vt')-\gamma F(\dot{\psi}_n). \tag{2}$$

It is important to note that if we ignore the ψ_n-vt' term and express $\dot{\psi}_n$ as a function of ψ_n Eq. (2) will become identical to the model equation (2.1) in [13].

In fact, since Eq. (2) is nondissipative, $F(\psi_n)$ is a pseudofriction force. As we shall see in the following, our nondissipative approach is justified by the consideration of dynamical events that correspond to a velocity weakening regime. We next assume [14] that Eq. (2) has a solution in the form of a nondissipative wave train described by a Jacobi elliptic function of parameter p,

$$\psi_n(z)=sn(z,p). \tag{3}$$

Here, $z=\Omega t-kna$, k and $\Omega=\omega/\omega_p$ are two constants. $c=\Omega/k$ represents the velocity of the wave. Now, following an inverse procedure, we first insert Eq. (3) in Eq. (2) in order to calculate the quantity $F(\dot{\psi}_n)$ as a function of ψ_n and express all terms ψ_n as a function of $\dot{\psi}_n$. Thus we have the following steps

$$\frac{d\psi_n}{dt'}=\Omega cn(z)dn(z)=\Omega\sqrt{(1-\psi_n^2)(1-p^2\psi^2)}, \tag{4}$$

$$\frac{d^2\psi_n}{dt'^2}=-2\Omega^2\psi_n[(1+p^2)-2p^2\psi_n^2], \tag{5}$$

$$\psi_{n+1}+\psi_{n-1}-2\psi_n=2\psi_n\left[\frac{\sqrt{(1-\xi^2)(1-p^2\xi^2)}}{1-p^2\xi^2\psi_n^2}-1\right]. \tag{6}$$

Here $\xi=sn(ka,p)$ represents a *discreteness parameter* as we shall see below. From Eq. (4), we can express ψ_n vs $\dot{\psi}_n$, in Eqs. (5) and (6). Under these conditions, after some lengthy but simple calculations we get

$$F(\dot{\psi}_n)=\frac{1}{\gamma}\sqrt{g}\left[2K\left(\frac{\sqrt{(1-\xi^2)(1-p^2\xi^2)}}{1-\xi^2g}-1\right)-1+\Omega^2\{(1+p^2)-2p^2g\}\right]-\frac{vt'}{\lambda\Omega},$$

$$\text{with}\quad g=\frac{(1+p^2)\pm\left[(1+p^2)^2-4p^2\left(1-\frac{\dot{\psi}_n^2}{\Omega^2}\right)\right]^{1/2}}{2p^2}. \tag{7}$$

Equation (2) with $F(\dot{\phi}_n)$ given by Eq. (7) admits Eq. (3) as an exact general periodic solution.

Now, in the context of this paper we restrict ourselves to the case $p=1$, that is, to a localized kink-shaped solution or solitonic shock front such as

$$\psi_n=sn(z,1)=\tanh(z), \tag{8}$$

with $\xi=sn(ka,1)=\tanh(ka)$. In this case, after setting $\phi_n=\psi_n/\lambda\Omega$, with $\lambda=(1-\xi^2)/\xi^2$, Eq. (7) reduces to

$$F(\dot{\phi}_n)=f_n-vt'/(\lambda\Omega)$$

with

$$f_n=\frac{1}{\lambda\Omega}\operatorname{sgn}(\dot{\phi}_n)\sqrt{1-\lambda|\dot{\phi}_n|}\times\left[2\lambda\Omega^2\left|\dot{\phi}_n\right|+2K\left(\frac{1}{1+|\dot{\phi}_n|}-1\right)-1\right]. \tag{9}$$

In order to ensure the symmetry of f_n with respect to zero, we have replaced $\dot{\phi}_n$ by $|\dot{\phi}_n|$. We have checked that this modification does not change the properties of the model. Finally, substituting Eq. (9) in Eq. (2) yields

$$\frac{d^2\phi_n}{dt^2}=K(\phi_{n+1}-2\phi_n+\phi_{n-1})-\phi_n-f_n. \tag{10}$$

The discrete equation of motion (10) with the nonlinear friction term f_n given by Eq. (9) admits

$$\phi_n=\frac{1}{\lambda\Omega}\tanh(\Omega t-kna), \tag{11}$$

as an exact solitonlike solution.

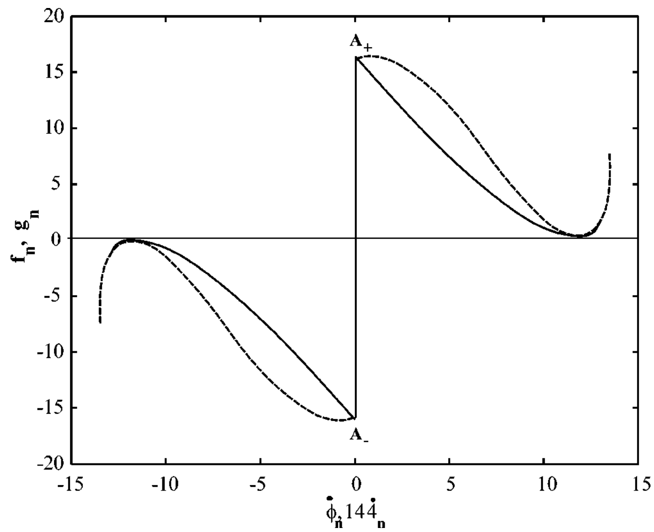

FIG. 2. Structure of the frictional force $f_n(\dot{\phi}_n)$ (continuous lines) and $g_n(\dot{\psi}_n)$ (dotted lines) for parameters $\xi = \tanh(2) \simeq 0.964$, $\omega = 1$, and $K = 0.1$. It presents the behavior of a simple velocity weakening friction law introduced qualitatively by Burridge and Knopoff and quantitatively by Carlson and Langer. In this figure all the quantities are given in normalized units.

This solitary wave solution corresponds to the dynamical regime. Since the so called loading term vt' is no more present in Eq. (10) our result implies to consider dynamical events (kinks) that occur at some fixed state of stress along the fault [15].

Nonlinear friction force. In Eq. (10), to each (Ω, k) corresponds a different force f_n and velocity c. We have checked numerically that the discrete kink or wave-front solution described by Eq. (11) can propagate stably, without emitting small radiations. In Fig. 2 the evolution of f_n (continuous curve) as a function of $\dot{\phi}_n$ is illustrated in the discrete regime (weak coupling), with $\xi = 0.964$, $\omega = 1$, and $K = 0.1$. f_n naturally presents the behavior of a simple velocity weakening friction law with instability or static friction discontinuity at zero velocity ($\dot{\phi}_n = 0$). Interestingly, by assuming a tanh-shaped wave-front solution we have obtained a frictional force f_n that exhibits the same type of behavior as for the friction force first introduced qualitatively by Burridge and Knopoff [3] and quantitatively by Carlson and Langer [4] and Carlson, Langer, and Shaw [6].

Evolution of an irregular initial state. It is clear that in a real system some degree of irregularity (spatial noise) should be present, and it is important to know whether complex behavior (seismicity) is due to (fault) heterogeneity or nonlinear dynamics.

With this in mind, we have investigated the evolution of an arbitrary distribution of N blocks whose positions are initialized by imposing a weak spatial perturbation (white noise). At time t_a (see Fig. 3), small-amplitude block motions appear. Then, after a while, we observe that some blocks can execute very large amplitude oscillations, much larger (about ten times) than the maximum amplitude of the original perturbation (noise) and can reach critical state of stress A_+ or A_- (see Fig. 2). As a result two kink pairs [(a) and (b)] can emerge (time t_1 in Fig. 3) and combine to give a kink and an antikink, that separate from each other. Furthermore, a stationary nonlinear localized mode appears and oscillates with time (t_2 and t_3 in Fig. 3). On the other hand, a discrete kink pair (c) can keep a constant profile and propagate to the right as a stable and robust pulse. These discrete wave fronts can travel freely along the lattice, because, stable solutions exist, as considered above.

We have also examined (not represented here) the evolution of an irregular initial state when the discreteness parameter is $\xi = \tanh(0.5) \simeq 0.462$, which corresponds to a strong coupling ($K = 10, \omega\psi = 1$) or continuum regime. In this case, to observe the emergence of at least one kink pair, we need to increase drastically (three times) the rms noise amplitude. If one further increases the rms amplitude no other kink pair emerges. Thus, our results show that the dynamical behavior of the continuum regime is very poor compared to the discrete regime where many kink pairs can be created.

Then, for a chain of $N = 200$ blocks, we have investigated the birth probability of at least one kink pair or one event, when the number of blocks initialized by a small perturbation (noise) is $N = 3$, 10, 50, while the other blocks are at rest. As shown in Fig. 4 the probability to obtain at least one event increases (continuous curve) strongly with the number of initialized blocks for the same initial noise amplitude. This result is not surprising since more energy is fed into the system.

III. BK MODEL WITH NONLINEAR COUPLING

In a real system (real fault), the interaction forces between blocks are not necessarily linear. They may be of the contact-type, like in a granular medium. With this assumption in mind we have investigated a BK model with nonlinear coupling [$q = 3$ in Eq. (2)],

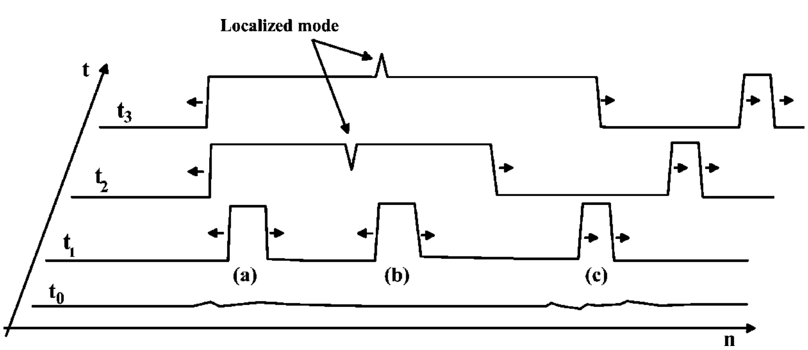

FIG. 3. Space-time evolution of an arbitrary initial state. Total number of blocks $N_T = 200$, number of excited blocks $N = 50$. Parameters: $\xi = \tanh(2) \simeq 0.964$, $\omega = 1$, and $K = 0.1$. At time t_0 small amplitude block motions appear. Then, three kink pairs (a), (b), and (c) emerge. They combine (time t_2) to give a propagating kink and antikink plus a stationary nonlinear localized mode that oscillates in time. The antikink pair (c), keeps a constant profile and propagate as a stable and robust pulse.

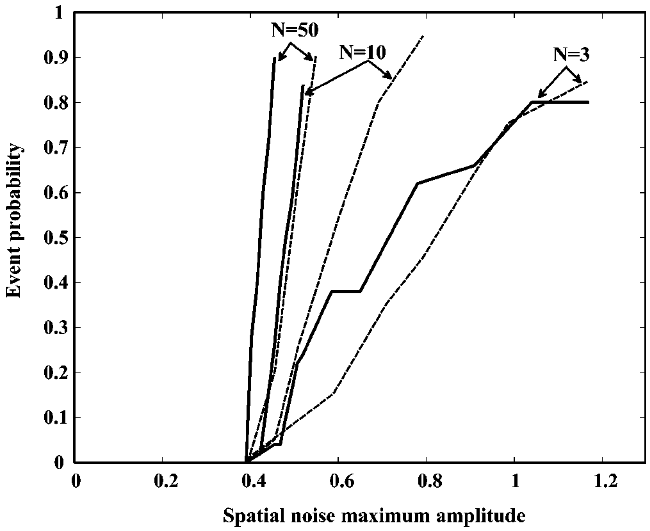

FIG. 4. Probability to obtain at least one event vs amplitude perturbation, for three different numbers of blocks initialized ($N=3$, $N=10$, $N=50$) on a chain of 200 blocks, others are at rest (solid and dashed lines corresponding respectively to linear coupling and nonlinear coupling of BK model).

$$\frac{d^2\psi n}{dt^2}=\omega_0^2[(\psi_{n+1}-\psi_n)^3-(\psi_n-\psi_{n-1})^3]-\psi_n-G(\dot{\psi}_n). \tag{12}$$

In Eq. (12) the power law (cubic)-type interaction suggests to assume that the shock solutions are strictly localized or compactlike [16–20], such as

$$\begin{aligned}\psi_n&=\sin(s),\quad \text{if } s\in[-\pi/2,+\pi/2],\\ \psi_n&=-1\quad \text{if } s\in[-\infty,-\pi/2],\\ \psi_n&=+1,\quad \text{if } s\in[+\pi/2,+\infty].\end{aligned} \tag{13}$$

Contrary to the linear model of Sec. II, where the tanh-shaped wave-front solution extends asymptotically to infinity, solution (13) has the advantage of taking into account the finite spatial extent of a wave front along the fault. Note that, like in Sec. II, we have set $vt'=0$, since we consider dynamical events (kinks) that occur at some fixed state of stress.

Proceeding as in Sec. II, after simple calculations we get,

$$G(\psi_n)=\alpha\psi_n+\beta\psi_n^3 \tag{14}$$

with

$$\begin{aligned}\alpha&=\omega^2-6\omega_0^2(\tau^3-\tau^2-r+1)-1,\\ \beta&=4\omega_0^2(1-3\tau^2+2\tau^3),\\ \tau&=\cos(ka).\end{aligned} \tag{15}$$

Since $\psi_n=\sqrt{1-\dot{\psi}_n^2/\omega^2}$, we finally obtain

$$G(\dot{\psi}_n)=g_n=\mathrm{sgn}(\dot{\psi}_n)\left(1-\frac{\dot{\psi}_n^2}{\omega^2}\right)\left[\alpha+\beta\left(1-\frac{\dot{\psi}_n^2}{\omega^2}\right)\right]. \tag{16}$$

As shown in Fig. 2 (dotted lines) the pseudofriction function g_n has the same general form as f_n.

Thus, with g_n given by Eq. (16) the modified discrete BK model admits exact compactlike solution given by Eq. (13). We have checked by numerical simulations, that for $\tau\leqslant\sqrt{2}/2$ (kink width $\geqslant 6a$) an exact solution given by Eq. (13) can propagate freely. For $\tau>\sqrt{2}/2$ (kink width $<6a$), discreteness effects appear: small nonlinear stationary oscillations are emitted during the kink propagation.

Next, we have investigated head on collisions between compactlike kink and antikink. For example, with $\tau\leqslant\sqrt{2}/2$ the collision (not represented here) between a kink and antikink of width $2a$ is pseudoelastic. The two kinks or shocks emerge with lower velocities, and a stationary nonlinear localized mode and small-amplitude nonlinear oscillations are created. Since the width of the localized mode is about $2a$, we have been unable to determine whether or not its shape is compactlike.

Evolution of an irregular initial state. Like for the linear BK model (see Sec. II), we have investigated the evolution of an initial noisy state, with the same kind of physical parameters [$\tau=\cos(2)\simeq-0.416$, $\omega=1$, $K=0.1$, $N_T=200$]. As shown in Fig. 5, small nonlinear oscillations appear in the beginning of the dynamics (time t_0, t_1, and t_2), then one observes the emergence (time t_3) of a kink pair dressed with oscillations. This pair (time t_4) evolves into a static compactlike antikink (in the presence of oscillations) and a traveling compactlike kink. In some simulations (not represented here) instead of static antikink one, one can observe a traveling one.

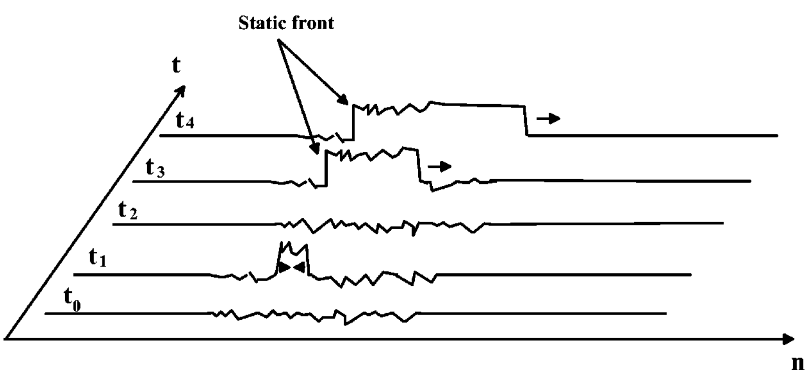

FIG. 5. Space-time evolution of an arbitrary initial state. Total number of blocks $N_T=200$, number of excited blocks $N=50$. Parameters $\tau=\cos(2)\simeq-0.416$, $\omega=1$, and $K=0.1$. At time t_0 small-amplitude blocks motions appear. Then, kink-antikink pair emerges at t_1, disappears at t_2, and reappears at t_3. The kink propagates freely to the right and the antikink remains immobile. Nevertheless, it may travel as observed in other simulations with the same parameters and irregular initial condition.

In the strong coupling or continuum regime [with discreteness parameter: $\tau = \cos(0.4) = 0.9$, $K = 10$, $\omega = 1$] our simulations (not represented here) show that no compactlike kink pair emerge from the initial state even for large rms noise amplitude.

Then, like in Sec. II, we have investigated the birth probability of at least one kink pair or one event, represented in Fig. 4 (dotted curves). For $N = 3$ or $N = 10$, the event probability does not increase so rapidly than for the linear BK model (continuous curves). For $N = 50$, the slopes tend to be the same.

IV. CONCLUSION

We have explored the dynamics of a discrete BK model with linear or nonlinear interactions between adjacent blocks. We have first shown that by properly choosing the analytical form of the discrete solitary wave solution of the model we can analytically calculate the form of the friction function. For linear coupling the kink solutions are tanh-shaped with infinitesimal wings extending to infinity. For nonlinear coupling the compactlike kink solutions are sine-shaped and strictly localized: with no wings. In both cases our analytical results show that the friction force naturally presents the behavior of a simple weakening friction law first introduced qualitatively by Burridge and Knopoff [3] and quantitatively by Carlson and Langer [4].

In the case of linear coupling our numerical simulations show that an irregular initial state evolves into several kink pairs that can separate, recombine or travel, plus localized modes and small linear oscillations that disperse with time, owing to dispersion. In the strong coupling (continuum) regime the dynamical behavior of the system is not so rich since only one kink pair emerges from the noisy initial state.

For nonlinear coupling one observes the emergence of compactlike kink pairs, which can separate into kinks traveling in opposite directions, and a background of robust incoherent nonlinear oscillations that survive with time. In the strong coupling or continuum regime no compactlike kink pair can emerge from the irregular initial state.

In both models discreteness is at the origin of a rich and complex dynamical behavior. It allows the creation of many narrow traveling kinks or shock fronts. Nonlinear coupling allows the existence of propagating kinks or shocks that are compactlike, that is, strictly localized. In the context of seismic events, the picture of narrow and compactlike shocks should be more realistic than the picture of solitonic shocks with asymptotic wings.

The model studied here is a simple caricature of a real system. Nevertheless, the results of this study, which takes into account the two important ingredients: discreteness and nonlinear coupling, should provide some guidance about how to make more realistic, predictive models of seismic phenomena.

[1] J. H. E. Cartwright, E. Hernandez-Garcia, and O. Piro, Phys. Rev. Lett. **79**, 527 (1997).
[2] C. B. Muratov, Phys. Rev. E **59**, 3847 (1999).
[3] R. Burridge and L. Knopoff, Bull. Seismol. Soc. Am. **57**, 3411 (1967).
[4] J. M. Carlson and J. S. Langer, Phys. Rev. Lett. **62**, 2632 (1989).
[5] P. Bak and C. Tang, J. Geophys. Res., [Space Phys.] **94**, 15 635 (1989).
[6] J. M. Carlson, J. S. Langer, and B. E. Shaw, Rev. Mod. Phys. **66**, 657 (1994).
[7] D. L. Turcotte, *Fractals and Chaos in Geology and Geophysics*, 2nd ed. (Cambridge University Press, Cambridge, England, 1997).
[8] J. Schmittbuhl, J. P. Villotte, and S. Roux, Europhys. Lett. **21**, 375 (1993).
[9] P. Espanol, Phys. Rev. E **50**, 227 (1994).
[10] M. A. Rubio and J. Galeano, Phys. Rev. E **50**, 1000 (1994).
[11] Z. L. Wu and Y. T. Chen, Nonlinear. Proc. Geophys. 121 (1998).
[12] J. R. Rice, J. Geophys. Res., [Space Phys.] **98**, 9885 (1993).
[13] J. C. Comte, P. Marquié, and M. Remoissenet, Phys. Rev. E **60**, 7484 (1999).
[14] M. H. Jensen, P. Per Bak, and A. Popielewicz, J. Phys. A **16**, 4369 (1983).
[15] J. S. Langer and C. Tang, Phys. Rev. Lett. **67**, 8 (1991); Interfaces **64**, 1 (1990).
[16] P. Rosenau and J. M. Hyman, Phys. Rev. Lett. **70**, 564 (1993).
[17] P. Rosenau, Phys. Rev. Lett. **73**, 1734 (1994).
[18] P. Rosenau, Phys. Rev. Lett. **252**, 297 (1999), and references therein.
[19] B. Dey and A. Khare, Phys. Rev. E **58**, R2741 (1998).
[20] M. Remoissenet, *Waves Called Solitons*, 3rd ed. (Springer-Verlag, Berlin, 1999).